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ON A NEW METHOD OF COMPUTING THE ROOTS OF BESSEL'S FUNCTIONS

BY WILLIAM MARSHALL

THE object of this paper is to show the application of a certain method to the computation of the roots of Bessel's functions. This method which was employed by me in my Doctor's Thesis* in the computation of the roots of the functions of the elliptic cylinder, and which is generally applicable in similar problems is essentially this: The coefficients of certain asymptotic expansions which are necessary in the computation, are not found by arithmetic processes, but are determined from recurrence formulas which arise when certain differential equations are formally satisfied. The results are of course not new. The roots of the Bessel's functions of order zero as determined from the semiconvergent expansion were first given by Stokes.† The roots of the Bessel's functions of order n are given by McMahan‡ and are found in Gray and Mathews.§

The ordinary Bessel's equation, namely

$$(1) \qquad \frac{d^2 J_n(x)}{dx^2} + \frac{1}{x}\frac{dJ_n(x)}{dx} + \left(1 - \frac{n^2}{x^2}\right) J_n(x) = 0,$$

can by means of the substitution $U = X^{\frac{1}{2}} J_n(x)$ be reduced to the form

$$(2) \qquad \frac{d^2 U}{dx^2} + \left(1 - \frac{n^2 - \frac{1}{4}}{x^2}\right) U = 0.$$

A solution of this in the neighborhood of $x = \infty$ is

$$(3) \qquad U = C \left\{ P \cos (a - x) + Q \sin (a - x) \right\},$$

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* Diss. Zürich, 1909, *American Journal of Mathematics*, vol. 31, no. 4, Oct., 1909.

† G. G Stokes, *Trans. Cambridge Phil. Soc.*, vol. 9, part I, or *Math. and Phys. Papers*, vol. 2, p. 329.

‡ ANNALS OF MATHEMATICS vol. 9, Jan., 1895.

§ *Treatise on Bessel's Functions*, p. 241.

where C and a are integration constants* and where P and Q represent the following asymptotic series:

$$(4)\quad \begin{cases} P = 1 - \dfrac{(4n^2-1)(4n^2-9)}{1.2(8x)^2} \\ \qquad\qquad + \dfrac{(4n^2-1)(4n^2-9)(4n^2-25)(4n^2-49)}{1.2.3.4\,(8x)^4} - \cdots \\ Q = \dfrac{4n^2-1}{8x} - \dfrac{(4n^2-1)(4n^2-9)(4n^2-25)}{1.2.3\,(8x)^3} + \cdots \end{cases}$$

If $C = \sqrt{\frac{2}{\pi}}$ and $a = \frac{2n+1}{4}\pi$, and if M and ψ are determined so that

$$(4') \qquad M\cos\psi = P \text{ and } M\sin\psi = Q,$$

then from (3)

$$(5) \qquad U = CM\cos(a - x - \psi),$$

and as $U = x^{\frac{1}{2}}J_n(x)$,

$$J_n(x) = \sqrt{\frac{2}{\pi x}}\,M\cos\left(\frac{2n+1}{4} - x - \psi\right),$$

and the roots of $J_n(x) = 0$ can be calculated from the relation

$$(6) \qquad \frac{2n+1}{4}\pi - x - \psi = -\frac{2k-1}{2}\pi (k = 1, 2, \cdots).$$

M and ψ have been calculated approximately and with great labor from the relations $M = \sqrt{P^2 + Q^2}$, $\psi = \tan^{-1}(Q/P)$ respectively. They may be obtained much more simply by the aid of equation (2) as follows. $U = CM\cos(a - x - \psi)$ will satisfy (2) whatever values may be given to C and a. Let $C = 1$, and $a = 0$. Then a solution of (2) is

$$(7) \qquad U = M\cos(\psi + x),$$

and M and ψ are to be determined so that (2) is satisfied. From (7) we obtain

$$(8) \qquad U' = \cos(\psi + x)M' - M(\psi' + 1)\sin(\psi + x),$$

---

* Gray and Mathews, l. c., p. 37.

$$U'' = \cos(\psi + x)\,M'' - 2M'\sin(\psi + x)(\psi' + 1) - M\cos(\psi + x)(\psi' + 1)^2 - M\sin(\psi + x)\psi''. \tag{9}$$

Substituting the values of U'' and U in (2) we obtain

$$M''\cos(\psi + x) - 2M'\sin(\psi + x)(\psi' + 1) - M\cos(\psi + x)(\psi' + 1)^2 - M\sin(\psi + x)\psi'' + M\cos(\psi + x)\left(1 - \frac{n^2 - \frac{1}{4}}{x^2}\right) = 0. \tag{10}$$

Since the coefficients of $\sin(\psi + x)$ and $\cos(\psi + x)$ must separately vanish, we obtain the two following differential equations for M and ψ, namely,

$$2M'(\psi' + 1) + M\psi'' = 0, \tag{11}$$

$$M'' - M(\psi' + 1)^2 + M\left(1 - \frac{n^2 - \frac{1}{4}}{x^2}\right) = 0. \tag{12}$$

Equation (11) integrates readily into

$$\psi' + 1 = \frac{k}{M^2}, \tag{13}$$

where k is the constant of integration, which may be determined from the fact that $\psi = \tan^{-1}\frac{Q}{P}$, whence $\psi' = \frac{PQ' - P'Q}{P^2 + Q^2}$. Therefore $k = 1$ and we have

$$\psi' + 1 = \frac{1}{M^2}. \tag{14}$$

Substituting this value of ψ' in (12) we have a differential equation for M, namely

$$M'' - \frac{1}{M^3} + M\left(1 - \frac{n^2 - \frac{1}{4}}{x^2}\right) = 0. \tag{15}$$

Now from (4') we see that the first term of M must be 1; we therefore assume as a solution of (15)

$$M = 1 + \frac{A_2}{x^2} + \frac{A_4}{x^4} + \frac{A_6}{x^6} + \cdots, \tag{16}$$

where the A's are undetermined constants. From (16) we obtain

$$M'' = \frac{3.2\,A_2}{x^4} + \frac{5.4\,A_4}{x^6} + \frac{7.6A_6}{x^8} + \cdots. \tag{17}$$

Also, by the Binomial Theorem,

$$\frac{1}{M^3} = 1 + \frac{a_2}{x^2} + \frac{a_4}{x^4} + \frac{a_6}{x^6} + \cdots, \tag{18}$$

where

$$\begin{aligned} a_2 &= -3A_2, \\ a_4 &= \frac{-4A_2\,a_2}{2} - 3A_4, \\ a_6 &= \frac{-5A_2a_4 - 7A_4a_2}{3} - 3A_6, \\ a_8 &= \frac{-6A_2a_6 - 8A_4a_4 - 10A_6a_2}{4} - 3A_8, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{19}$$

or in terms of the A's alone

$$\begin{aligned} a_2 &= -3A_2, \\ a_4 &= 6A_2^2 - 3A_4, \\ a_6 &= -10A_2^3 + 12A_2A_4 - 3A_6, \\ a_8 &= 15A_2^4 - 30A_2^2A_4 + 12A_2A_6 + 6A_4^2 - 3A_8, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{20}$$

Substituting now in (15) the value of M from (16), M'' from (17), and $1/M^3$ from (18), we have

$$\begin{aligned} &\frac{3.2\,A_2}{x^4} + \frac{5.4\,A_4}{x^6} + \frac{7.6\,A_6}{x^8} + \cdots \\ &+ \frac{A_2}{x^2} + \frac{A_4}{x^4} + \frac{A_6}{x^6} + \frac{A_8}{x^8} + \cdots \\ &- \frac{n^2 - \frac{1}{4}}{x^2} - \frac{(n^2 - \frac{1}{4})A_2}{x^4} - \frac{)n^2 - \frac{1}{4})A_4}{x^6} - \frac{(n^2 - \frac{1}{4})}{8^8}A_6 - \cdots \\ &- \frac{a_2}{x^2} - \frac{a_4}{x^4} - \frac{a_6}{x^6} - \frac{a_8}{x^8} - \cdots = 0. \end{aligned} \tag{21}$$

Equating to zero the coefficients of like powers of x and making use of the relations (19) we have

$$
\begin{aligned}
&4A_2 = n^2 - \tfrac{1}{4},\\
&4A_4 = (n^2 - \tfrac{1}{4} - 3.2)A_2 - \frac{4A_2a_2}{2},\\
(22)\qquad &4A_6 = (n^2 - \tfrac{1}{4} - 5.4)A_4 - \frac{5A_2a_4 + 7A_4a_2}{3},\\
&4A_8 = (n^2 - \tfrac{1}{4} - 7.6)A_6 - \frac{6A_2a_6 + 8A_4a_4 + 10A_6a_2}{4},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

Formulas (22) then enable us to compute the coefficients in the expansion of $M = \sqrt{P^2 + Q^2}$ without serious labor. These formulas are, in terms of the A's,

$$
\begin{aligned}
&4A_2 = n^2 - \frac{1}{4},\\
&4A_4 = \left(n^2 - \frac{25}{4}\right) A_2 + 6A_2^2,\\
(23)\qquad &4A_6 = \left(n^2 - \frac{81}{4}\right) A_4 - 10A_2^3 + 12A_2A_4,\\
&4A_8 = \left(n^2 - \frac{169}{4}\right) A_6 + 15A_2^4 - 30A_2^2A_4 + 12A_2A_6 + 6A_4^2,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

or in terms of n alone,

$$
\begin{aligned}
&A_2 = \frac{1}{2^4}(4n^2 - 1),\\
&A_4 = \frac{1}{2^9}(4n^2 - 1)(20n^2 - 53),\\
(24)\qquad &A_6 = \frac{1}{2^{13}}(4n^2 - 1)(240n^4 - 2488n^2 + 4477),\\
&A_8 = \frac{1}{2^{19}}(4n^2 - 1)(12480n^6 - 322192n^4 + 2023460n^2 - 3066403)\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

or, if we set for shortness $4n^2 = m$,

$$
\begin{aligned}
A_2 &= \frac{1}{2^4}(m-1), \\
A_4 &= \frac{1}{2^9}(m-1)(5m-53), \\
A_6 &= \frac{1}{2^{13}}(m-1)(15m^2-622m+4447), \qquad (25) \\
A_8 &= \frac{1}{2^{19}}(m-1)(195\,m^3-20137m^2+505865m-3066403), \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

From these coefficients in the expansion of M, we may readily compute those in the development of $\psi = \tan^{-1} Q/P$. We have from (14)

$$
(26) \qquad \psi' = \frac{1}{M^2} - 1.
$$

Then ψ' is of the form

$$
(27) \qquad \psi' = \frac{c_2}{x^2} + \frac{c_4}{x^4} + \frac{c_6}{x^6} + \cdots\cdot
$$

Integrating, and setting the constant of integration equal to zero, since $\tan \psi = Q/P$, we have

$$
(28) \qquad \psi = \frac{-C_2}{x} - \frac{C_4}{3x^3} - \frac{C_6}{5x^5} - \frac{C_8}{7x^7} - \cdots\cdot,
$$

where the C's are simply the coefficients in the expansion of $1/M^2$ by the Binomial theorem, that is,

$$
\begin{aligned}
C_2 &= -2A_2, \\
2C_4 &= -3A_2C_2 - 4A_4, \\
3C_6 &= -4A_2C_4 - 5A_4C_2 - 6A_6, \qquad (29) \\
4C_8 &= -5A_2C_6 - 6A_4C_4 - 7A_6C_2 - 8A_8, \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

or, in terms of the A's,

$$
(30)\qquad
\begin{aligned}
C_2 &= -\ 2A_2,\\
C_4 &= \quad 3A_2^2 - 2A_4,\\
C_6 &= -\ 4A_2^3 + 6A_2A_4 - 2A_6,\\
C_8 &= \quad 5A_2^4 - 12A_2^2A_4 + 6A_2A_6 + 3A_4^2 - 2A_8,\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

These coefficients in the expansion of ψ are used in the computation of the roots of $J_n(x)$ in the following manner: an asymptotic value of $J_n(x)$ is *

$$(31)\qquad J_n(x) = \sqrt{\frac{2}{\pi x}}\left[P\cos\left(\frac{2n+1}{4}\pi - x\right) + Q\sin\left(\frac{2n+1}{4}\pi - x\right)\right].$$

This may be written

$$(32)\qquad J_n(x) = \sqrt{\frac{2}{\pi x}}\,M\cos\left(\frac{2n+1}{4}\pi - x - \psi\right),$$

which vanishes when

$$(33)\qquad \frac{2n+1}{4}\pi - x - \psi = -\frac{2k-1}{2}\pi,$$

that is, when

$$(34)\qquad x = \frac{\pi}{4}(2n - 1 + 4k) - \psi,$$

where k is any integer.

If we put for shortness $\frac{\pi}{4}(2n - 1 + 4k) = \beta$,

then the values of x for which $J_n(x)$ vanishes are found by solving the equation

$$(35)\qquad x = \beta - \psi,$$

or

$$(36)\qquad x = \beta + \frac{C_2}{x} + \frac{C_4}{3x^3} + \frac{C_6}{5x^5} + \cdots.$$

Equation (36) is best solved by the method of successive approximations. The successive approximations are

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* Gray and Mathews, *Treatise on Bessel's Functions*, p. 40.

$$(37)\left\{\begin{aligned}
x_1 &= \beta_1,\\
x_2 &= \beta + \frac{C_2}{\beta},\\
x_3 &= \beta + \frac{C_2}{\beta} + \frac{\frac{C_4}{3} - C_2^2}{\beta^3},\\
x_4 &= \beta + \frac{C_2}{\beta} + \frac{\frac{C_4}{3} - C_2^2}{\beta^3} + \frac{2C_2^3 - \frac{4}{3}C_2C_4 + \frac{C_6}{5}}{\beta^5},\\
x_5 &= \beta + \frac{C_2}{\beta} + \frac{\frac{C_4}{3} - C_2^2}{\beta^3} + \frac{\frac{C_6}{5} - \frac{4}{3}C_2C_4 + 2C_2^3}{\beta^5}\\
&\quad + \frac{\frac{C_8}{7} - 5C_2^4 - \frac{6}{5}C_2C_6 + 5C_2^2C_4 - \frac{C_4^2}{3}}{\beta^7}.
\end{aligned}\right.$$

Expressing these coefficients of β^{-1}, $\beta^{-3}\cdots$ in terms of m by means of the relations (25) and (30) we have for the kth root of $J_n(x)$,

$$(38)\qquad \begin{aligned} x_k = \beta - \frac{m-1}{2^3\beta} - \frac{(m-1)}{3\cdot 2^7}\cdot\frac{(7m-31)}{\beta^3} \\ - \frac{(m-1)}{15.2^{10}}\cdot\frac{(83m^2 - 982m + 3779)}{\beta^5} \\ - \frac{(m-1)}{105.2^{15}}\cdot\frac{(6949m^3 - 153855m^2 + 1585743m - 6277237)}{\beta^7} - \cdots\end{aligned}$$

where $$\beta = \frac{\pi}{4}(2n + 4k - 1),$$

and $$m = 4n^2.$$

This result agrees with that given by McMahan;* there is however a slight error in the formula as given by Gray and Mathews,† due to the dropping of a figure in the coefficient of β^{-7}.

PURDUE UNIVERSITY.
OCTOBER, 1909.

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* ANNALS OF MATHEMATICS, ser. 2, vol. 9, p. 25.

† Loc. cit., p. 241.